

Small coupling limit and multiple solutions to the Dirichlet Problem for Yang Mills connections in 4 dimensions - Part III

Takeshi Isobe ^{*} and Antonella Marini [†]

Abstract

In this paper, the third of this series, we prove that the spaces $\mathcal{A}_k^{*,p}(A_0; \mathbf{q})$ and $\mathcal{B}_{0,k}^{*,p}(A_0; \mathbf{q})$ which contain L_k^p -approximate solutions to the Dirichlet problem for the ϵ -Yang Mills equations on a four dimensional disk B^4 , carry a natural manifold structure (more precisely a natural structure of Banach bundle), for $p(k+1) > 4$. All results apply also if B^4 is replaced by a general compact manifold with boundary, and $SU(2)$ is replaced by any compact Lie group. We also construct bases for the tangent space to the space of approximate solutions, thus showing that this space is 8-dimensional for ϵ sufficiently small, and prove some technical results used in Parts I and II for the proof of the existence of multiple solution and, in particular, non-minimal ones, for this non-compact variational problem.

1 Introduction

We consider connections A over principal bundles P over the four dimensional disk, with fiber isomorphic to $SU(2)_\epsilon$, for $\epsilon > 0$, where $SU(2)_\epsilon = SU(2)$ as a set, but it is endowed with the ϵ -deformed multiplication, i.e., its Lie algebra is $(\mathfrak{su}(2)_\epsilon, [\cdot, \cdot]_\epsilon) := (\mathfrak{su}(2), \epsilon[\cdot, \cdot])$, where $[\cdot, \cdot]$ is the ordinary Lie bracket on $\mathfrak{su}(2)$. We recall that for a given smooth boundary value A_0 , the Dirichlet problem for the ϵ -Yang Mills equations is obtained via a variational principle from the $SU(2)_\epsilon$ -Yang Mills functional

$$\mathcal{YM}_\epsilon(A) = \int_{B^4} |F_A^\epsilon|^2 dx, \quad (1.1)$$

where $F_A^\epsilon = dA + \frac{1}{2}[A, A]_\epsilon := dA + \frac{\epsilon}{2}[A, A]$, and consists of the system

$$(\mathcal{D}_\epsilon) \quad \begin{cases} d_A^{*\epsilon} F_A^\epsilon = 0 & \text{in } B^4 \\ \iota^* A \sim A_0 & \text{at } \partial B^4, \end{cases}$$

where, $\iota : \partial B^4 \rightarrow \overline{B}^4$ is the inclusion, the symbol \sim stands for gauge equivalence via a gauge transformation that extends smoothly to the interior, and $d_A^{*\epsilon} := *d* + *[A, *]_\epsilon := *d* + \epsilon*[A, *]$, where $*$ is the Hodge star operator with respect to the flat metric on \mathbb{R}^4 .

An absolute minimum, say \underline{A}_ϵ , for the Yang-Mills functional is known to exist by [6]. Moreover, in [3], it is shown that the space of connections with boundary value A_0 , denoted by $\mathcal{A}(A_0)$, has countable connected components, i.e., $\mathcal{A}(A_0) = \bigsqcup_{j=-\infty}^{\infty} \mathcal{A}_j(A_0)$, where $\mathcal{A}_j(A_0)$ is the space of connections with relative 2nd Chern number with respect to \underline{A}_ϵ equal to j , and that there

^{*}Tokyo Institute of Technology; email: isobe.t.ab@m.titech.ac.jp

[†]University of L'Aquila / Yeshiva University; email: marini@dm.univaq.it

always exists a minimizer in $\mathcal{A}_{+1}(A_0)$ (or $\mathcal{A}_{-1}(A_0)$) if A_0 is not flat. In [4, 5] we have found solutions to (\mathcal{D}_ϵ) in $\mathcal{A}_{+1}(A_0)$, by first constructing approximate solutions, for small values of the parameter $\epsilon > 0$. However, some of our proofs utilize technical results proved in the present paper.

Throughout this paper, we assume $\lambda_0, D_1, D_2, \epsilon, \mathbf{q}$ be fixed, with $0 < 2\lambda_0 < d_0, 0 < D_1 < D_2$, $\mathbf{q} := (p, [g], \lambda) \in \mathcal{P}(d_0, \lambda_0; D_1, D_2; \epsilon) := \{\mathbf{q} := (p, [g], \lambda) \in \mathcal{P}(d_0, \lambda_0) : D_1\epsilon < \lambda^2 < D_2\epsilon\}$, where $\mathcal{P}(d_0, \lambda_0) := B_{1-d_0}^4 \times SO(3) \times (0, \lambda_0)$ is the parameter space used in [4] in the gluing procedure to construct approximate solutions to the ϵ -Dirichlet problem, that look like the connected sum of $\underline{A}_\epsilon \# \frac{1}{\epsilon}(1\text{-bubble})$.

Although all our arguments apply to the general case of M , any smooth compact manifold with boundary, and G , any compact Lie group, we focus on $M = B^4$ and $G = SU(2)$.

We recall (cf. [4]) that, for $\mathbf{q} := (p, g, \lambda)$ fixed,

$$A(\mathbf{q}) = \begin{cases} (1 - \beta_{\lambda,p})\underline{A}_\epsilon + \frac{1}{\epsilon}\beta_{\lambda/4,p}gI_{\lambda,p}^2g^{-1} + \frac{1}{\epsilon}(1 - \beta_{\lambda/4,p})gPI_{\lambda,p}^2g^{-1} & \text{in } B^4 \setminus \{p\} \\ \frac{1}{\epsilon}gI_{\lambda,p}^1g^{-1} & \text{in } B_{\lambda/4}^4(p), \end{cases} \quad (1.2)$$

where $I_{\lambda,p}^1(x) = \text{Im} \frac{(x-p)d\bar{x}}{\lambda^2 + |x-p|^2}$ in $B^4(p) := \{x \in \mathbb{R}^4 : |x-p| < 1\}$, together with $I_{\lambda,p}^2(x) = \text{Im} \frac{\lambda^2(\bar{x}-\bar{p})dx}{|x-p|^2(\lambda^2 + |x-p|^2)}$ in $\mathbb{R}^4 \setminus \{p\}$ compose the 1-instanton solution to the Yang Mills equations on \mathbb{R}^4 , $\beta(x) = \beta(|x|) \in C_0^\infty(\mathbb{R}^4)$ is such that $\beta = 1$ for $|x| \leq 1$, $\beta(x) = 0$ for $|x| \geq 2$ and $0 \leq \beta(x) \leq 1$, and $\beta_{\lambda,p}(x) := \beta(\lambda^{-1}(x-p))$.

These connections form the space

$$\mathcal{N}(d_0, \lambda_0) := \{A(\mathbf{q}) : \mathbf{q} \in \mathcal{P}(d_0, \lambda_0)\}, \quad (1.3)$$

of approximate solutions to (\mathcal{D}_ϵ) with relative second Chern class equal to $+1$.

Let $g_{12,p}(x) = \frac{x-p}{|x-p|}$ be the transition maps for the 1-instanton solution defined above. The connections $A(\mathbf{q})$ live on the bundles

$$P(\mathbf{q}) := \left(B_{\lambda/4}^4(p), B^4 \setminus \{p\}, g g_{12,p} g^{-1} \right). \quad (1.4)$$

(Notice that $P(p, g, \lambda) = P(p, -g, \lambda)$ and $A(p, g, \lambda) = A(p, -g, \lambda)$, that's why we take $g \in SU(2)/\{\pm 1\} \cong SO(3)$).

We define $\mathcal{A}^*(A_0; \mathbf{q})$ as the space of all connections over $P(\mathbf{q})$ satisfying $\iota^*A \sim A_0$ on ∂B^4 , via a gauge transformation $g \in \mathcal{G}^*(\mathbf{q})$ (the group of all smooth gauge transformations at ∂B^4 such that $g((1, 0, 0, 0)) = \mathbf{1}$), which extends smoothly to the interior, and $\mathcal{B}(A_0; \mathbf{q})^* := \mathcal{A}^*(A_0; \mathbf{q})/\mathcal{G}^*(\mathbf{q})$. We define also the Sobolev counterparts of these spaces: $\mathcal{A}_k^{*,p}(A_0; \mathbf{q})$, as the space of all connections over $P(\mathbf{q})$ satisfying $\iota^*A \sim A_0$ on ∂B^4 , via a gauge transformation $g \in \mathcal{G}_{k+1-1/p}^{*,p}(\mathbf{q})$ (the group of all gauge transformations in $L_{k+1-1/p}^p(\partial B^4)$ such that $g((1, 0, 0, 0)) = \mathbf{1}$), which admits an L_{k+1}^p extension to the interior, and $\mathcal{B}_k^{*,p}(A_0; \mathbf{q}) := \mathcal{A}_k^{*,p}(A_0; \mathbf{q})/\mathcal{G}_{k+1}^{*,p}(\mathbf{q})$.

We assume $p(k+1) > 4$ throughout the present paper.

2 A manifold structure for $\mathcal{A}_k^{*,p}(A_0; \mathbf{q})$ and $\mathcal{B}_k^{*,p}(A_0; \mathbf{q})$

The space of all connections on a principal bundle P is an affine space and carries a natural differentiable structure, while $\mathcal{A}^*(A_0; \mathbf{q})$ is not an affine space and it is non-trivial to show that it carries a natural differentiable structure. The purpose of this section is to show that $\mathcal{A}^*(A_0; \mathbf{q})$, its quotient $\mathcal{B}^*(A_0; \mathbf{q})$, and, more in particular, their Sobolev counterparts, possess a natural differentiable structure.

We start with $\mathcal{A}_k^{*,p}(A_0; \mathbf{q})$. In what follows, we omit the dependence on \mathbf{q} in the notation.

Proposition 2.1 *The space of connections $\mathcal{A}_k^{*,p}(A_0)$ carries a natural structure of Banach bundle over $\mathcal{G}_{k+1-1/p}^{*,p}$. More precisely, for $A \in \mathcal{A}_k^{*,p}(A_0)$ with $\iota^*A = g^*A_0$ on ∂B^4 , for some $g \in \mathcal{G}_{k+1-1/p}^{*,p}$, define $\pi : \mathcal{A}_k^{*,p}(A_0) \rightarrow \mathcal{G}_{k+1-1/p}^{*,p}$ by $\pi(A) = g$. The map π is a well-defined projection, and $\pi : \mathcal{A}_k^{*,p}(A_0) \rightarrow \mathcal{G}_{k+1-1/p}^{*,p}$ is a vector bundle with fiber isomorphic to $L_{0T,k}^p(T^*B^4 \otimes \text{Ad}(P))$.*

Proof: By the Sobolev embedding $L_{k+1-1/p}^p(\partial B^4) \subset C^0(\partial B^4)$ for $p(k+1) > 4$, $\mathcal{G}_{k+1-1/p}^{*,p}$ is a Banach Lie group. Since $\mathcal{G}_{k+1-1/p}^{*,p}$ acts freely on $\mathcal{A}_k^{*,p}(A_0)$ (the restriction of g to the boundary is uniquely determined), the map $\pi : \mathcal{A}_k^{*,p}(A_0) \rightarrow \mathcal{G}_{k+1-1/p}^{*,p}$ is well-defined.

The group $\mathcal{G}_{k+1-1/p}^{*,p}$ possesses a natural manifold structure. In fact, the space

$$L_{k+1-1/p}^{*,p}(\text{Ad}(P|_{\partial B^4})) := \{u \in L_{k+1-1/p}^p(\text{Ad}(P|_{\partial B^4})) : u((1, 0, 0, 0)) = 0\}$$

is a closed subspace of the Banach space $L_{k+1-1/p}^p(\text{Aut}(P|_{\partial B^4}))$, which is isomorphic to the Lie algebra of $\mathcal{G}_{k+1-1/p}^{*,p}$. For $u \in L_{k+1-1/p}^{*,p}(\text{Ad}(P|_{\partial B^4}))$, and fixed $s \in \mathcal{G}_{k+1-1/p}^{*,p}$, define $(\text{Exp}_s u)(x) = s(x) \exp u(x)$, where “exp” denotes the exponential map for the Lie group $SU(2)$. A local chart of $\mathcal{G}_{k+1-1/p}^{*,p}$ at s is then constructed as follows. Set

$$U := \{u \in L_{k+1-1/p}^{*,p}(\text{Ad}(P|_{\partial B^4})) : \|u\|_{k+1-1/p,p} < \sigma\}, \quad (2.1)$$

where $0 < \sigma < \frac{r}{3}$, where r is the injectivity radius of $SU(2)$. Let $\tilde{U}_s := \text{Exp}_s(U)$. With this definition, \tilde{U}_s is a neighborhood of s in $\mathcal{G}_{k+1-1/p}^{*,p}$ and $\text{Exp}_s : U \rightarrow \tilde{U}_s$ yields a local chart near s . See [2] for more details.

We construct the ‘extension operator’ $T : \mathcal{G}_{k+1-1/p}^{*,p} \rightarrow \mathcal{G}_{k+1}^{*,p}$, in such a way that for any $s \in \mathcal{G}_{k+1}^{*,p}$, there exists a neighborhood U (as in (2.1)) of ι^*s in $\mathcal{G}_{k+1-1/p}^{*,p}$ such that its restriction $T|_U =: T_s : U \rightarrow \mathcal{G}_{k+1}^{*,p}$ is smooth and satisfies $T_s(\iota^*s) = s$.

We take a smooth connection which extends A_0 to P . For simplicity, we also denote this connection by A_0 . Set $\Delta_{A_0} = \nabla_{A_0}^* \nabla_{A_0} : L_{k+1}^p(\text{Ad}(P)) \rightarrow L_{k-1}^p(\text{Ad}(P))$. Let $U \subset \mathcal{G}_{k+1-1/p}^{*,p}$ be a small neighborhood of ι^*s . We first define the linear operator $T_0 : U \rightarrow L_{k+1}^{*,p}(\text{Ad}(P))$ as follows (here we identify locally $\mathcal{G}_{k+1-1/p}^{*,p}$ with its Lie algebra $L_{k+1-1/p}^{*,p}(\text{Ad}(P|_{\partial B^4}))$ via the exponential map): for given $\xi \in U$, $u = T_0(\xi)$ is the unique solution of $\Delta_{A_0} u = 0$ in B^4 with boundary value $u = \xi$ on ∂B^4 . Elliptic estimates yield smoothness of T_0 . Moreover, $\|T_0(\xi)\|_{k+1,p} \leq C\|\xi\|_{k+1-1/p,p} \leq C\sigma$, for some $C > 0$ independent of ξ . Thus, for small σ , $T_s(\xi) := \text{Exp}_s(T_0(\xi))$ is well-defined, smooth with respect to s and ξ , and $T_s(0) = s$.

We next define local charts for $\mathcal{A}_k^{*,p}(A_0)$. Let U be as in (2.1). We observe that any $A \in \mathcal{A}_k^{*,p}(A_0)$ can be written as $A = g^*(A_0 + \alpha)$, for some $g \in \mathcal{G}_{k+1}^{*,p}$ and some $\alpha \in L_{0T,k}^p(T^*B^4 \otimes \text{Ad}(P))$. Hence, for $s \in \mathcal{G}_{k+1-1/p}^{*,p}$, we can define the map

$$\begin{aligned}\Phi_s : U \times L_{0T,k}^p(T^* \otimes \text{Ad}(P)) &\rightarrow \mathcal{A}_k^{*,p}(A_0), \\ \Phi_s(u, \alpha) &= T_s(u)^*(A_0 + \alpha).\end{aligned}$$

In order to complete this proof we need the following two lemmas.

Lemma 2.1 Φ_s is one to one.

Proof: One has that $\Phi_s(u, \alpha) = \Phi_s(v, \beta)$ for $(u, \alpha), (v, \beta) \in U \times L_{0T,k}^p(T^*B^4 \otimes \text{Ad}(P))$ if and only if

$$T_s(u)^*A_0 + T_s(u)^{-1}\alpha T_s(u) = T_s(v)^*A_0 + T_s(v)^{-1}\beta T_s(v).$$

The restriction of this formula to ∂B^4 yields $\iota^*T_s(u)^*A_0 = \iota^*T_s(v)^*A_0$, and, since the $\mathcal{G}_{k+1-1/p}^{*,p}$ -action is free, $\iota^*T_s(u) = \iota^*T_s(v)$ on ∂B^4 . Thus, $\text{Exp}_s u = \text{Exp}_s v$, by definition of T_s . From this, $u = v$ and $\alpha = \beta$. \square

Lemma 2.2 $\{\Phi_s, U \times L_{0T,k}^p(T^*B^4 \otimes \text{Ad}(P))\}_{s \in \mathcal{G}_{k+1-1/p}^{*,p}}$ is a differentiable vector bundle structure for $\mathcal{A}_k^{*,p}(A_0)$.

Proof: Let $s, t \in \mathcal{G}_{k+1-1/p}^{*,p}$ with $\|s - t\|_{k+1-1/p, p} < \epsilon$ for some small $\epsilon > 0$. Here and in what follows, we replace ι^*s by s in our notation.

We need to show that $\Phi_s^{-1} \circ \Phi_t : U \times L_{0T,k}^p(T^*B^4 \otimes \text{Ad}(P)) \rightarrow U \times L_{0T,k}^p(T^*B^4 \otimes \text{Ad}(P))$ is smooth for small positive ϵ . Setting $\Phi_s^{-1} \circ \Phi_t(u, \alpha) = (v, \beta)$, for $(u, \alpha), (v, \beta) \in U \times L_{0T,k}^p(T^*B^4 \otimes \text{Ad}(P))$, one has $v = \text{Exp}_s^{-1} \circ \text{Exp}_t u$ (cf. Lemma 2.1). Then, β is given by

$$\beta = T_s(\text{Exp}_s^{-1} \circ \text{Exp}_t u)(T_t(u)^*A_0 + T_t(u)^{-1}\alpha T_t(u) - T_s(\text{Exp}_s^{-1} \circ \text{Exp}_t u)^*A_0) T_s(\text{Exp}_s^{-1} \circ \text{Exp}_t u)^{-1}.$$

Thus, v and β depend smoothly on u and α . This completes the proof. \square

Completion of the proof of Proposition 2.1: We observe that for $s \in \mathcal{G}_{k+1-1/p}^{*,p}$

$$\begin{aligned}\pi^{-1}(s) &= \{A \in \mathcal{A}_k^{*,p}(A_0) : \iota^*A = s^*A_0\} \\ &= \{A = \tilde{s}^*(A_0 + \alpha) : \alpha \in L_{0T,k}^p(T^*B^4 \otimes \text{Ad}(P))\} \cong L_{0T,k}^p(T^*B^4 \otimes \text{Ad}(P)),\end{aligned}$$

where $\tilde{s} \in \mathcal{G}_{k+1}^{*,p}$ satisfies $\iota^*\tilde{s} = s$ and

$$\begin{aligned}\pi^{-1}(\tilde{U}_s) &= \{A \in \mathcal{A}_k^{*,p}(A_0) : \iota^*A = g^*A_0, g \in \tilde{U}_s\} \\ &= \{A = T_s(u)^*(A_0 + \alpha) : u \in U, \alpha \in L_{0T,k}^p(T^*B^4 \otimes \text{Ad}(P))\} \cong \tilde{U}_s \times L_{0T,k}^p(T^*B^4 \otimes \text{Ad}(P)).\end{aligned}$$

Combining these results with Lemma 2.2, one sees that $\pi : \mathcal{A}_k^{*,p}(A_0) \rightarrow \mathcal{G}_{k+1-1/p}^{*,p}$ is a vector bundle with fiber isomorphic to $L_{0T,k}^p(T^*B^4 \otimes \text{Ad}(P))$. This completes the proof. \square

The following proposition holds for the quotient space $\mathcal{B}_k^{*,p}(A_0) = \mathcal{A}_k^{*,p}(A_0)/\mathcal{G}_{k+1}^{*,p}$:

Proposition 2.2 *The space $\mathcal{B}_k^{*,p}(A_0)$ defined in §1 has a differentiable manifold structure. Moreover, the canonical projection $\mathcal{A}_k^{*,p}(A_0) \rightarrow \mathcal{B}_k^{*,p}(A_0)$ yields a principal $\mathcal{G}_{k+1}^{*,p}$ -bundle.*

Proof: The argument is standard (cf. [1], [2]). One first seeks a candidate for the slice in $\mathcal{A}_k^{*,p}(A_0)$ at a given $A_1 \in \mathcal{A}_k^{*,p}(A_0)$.

For $\xi \in L_{k+1}^{*,p}(\text{Ad}(P))$ and $t \in \mathbb{R}$ with $|t|$ small, define $g_t := \text{Exp}_1(t\xi) \in \mathcal{G}_{k+1}^{*,p}$. There exists $g_0 \in \mathcal{G}_{k+1}^{*,p}$ and $\alpha_0 \in L_{0T,k}^p(T^*B^4 \otimes \text{Ad}(P))$ such that $A_1 = g_0^*(A_0 + \alpha_0)$. From

$$\begin{aligned} g_t^* A_1 &= (\text{Exp}_{g_0}(t\xi))^*(A_0 + \alpha_0) \\ &= T_{g_0}(t\xi)^*[A_0 + \alpha_0 + T_{g_0}(t\xi)(\text{Exp}(t\xi)^*(A_0 + \alpha_0) - T_{g_0}(t\xi)^*(A_0 + \alpha_0))T_{g_0}(t\xi)^{-1}], \end{aligned}$$

one has

$$\Phi_{g_0}^{-1}(g_t^* A_1) = (t\xi|_{\partial B^4}, \alpha_0 + T_{g_0}(t\xi)(\text{Exp}_{g_0}(t\xi)^*(A_0 + \alpha_0) - T_{g_0}(t\xi)^*(A_0 + \alpha_0))T_{g_0}(t\xi)^{-1}).$$

Thus a tangent vector to the orbit $\{g_t^* A_1\}_t$ at A_1 is given by

$$\left. \frac{d}{dt} \right|_{t=0} \Phi_{g_0}^{-1}(g_t^* A_1) = (\xi|_{\partial B^4}, g_0(d_{A_1}\xi - d_{A_1}T_0(\xi))g_0^{-1}).$$

Therefore, a vector $(u, \alpha) \in L_{k+1-1/p}^{*,p}(\text{Ad}(P|_{\partial B^4})) \times L_{0T,k}^p(T^*B^4 \otimes \text{Ad}(P))$ in the slice at A_1 must satisfy

$$(\xi|_{\partial B^4}, u)_{L^2(\partial B^4)} + (g_0(d_{A_1}(\xi - T_0(\xi))g_0^{-1}, \alpha)_{L^2(B^4)} = 0 \quad \forall \xi \in L_{k+1}^{*,p}(\text{Ad}(P)). \quad (2.2)$$

By taking $\xi|_{\partial B^4} = 0$, one has $T_0(\xi) = 0$, and (2.2) yields

$$(g_0(d_{A_1}\xi)g_0^{-1}, \alpha)_{L^2(B^4)} = 0, \quad (2.3)$$

thus

$$d_{A_0+\alpha_0}^* \alpha = 0 \quad \text{in } M. \quad (2.4)$$

On the other hand, (2.2), (2.4) yield $(\xi|_{\partial B^4}, u)_{L^2(\partial B^4)} = 0$ for all $\xi \in L_{k+1-1/p}^{*,p}(\text{Ad}(P|_{\partial B^4}))$, thus $u = 0$. Therefore, a suitable candidate for the slice at A_1 is

$$N = \{(0, \alpha) \in L_{k+1-1/p}^{*,p}(\text{Ad}(P|_{\partial B^4})) \times L_{0T,k}^p(T^*B^4 \otimes \text{Ad}(P)) : d_{A_0+\alpha_0}^* \alpha = 0\}.$$

We must now prove rigorously that $\mathcal{A}_k^{*,p}(A_0)$ is indeed locally diffeomorphic to $N \times \mathcal{G}_{k+1}^{*,p}$, i.e., that there exists a neighborhood of $A_1 \in \mathcal{A}_k^{*,p}(A_0)$, say $\mathcal{O}(A_1)$, and $g : \mathcal{O}(A_1) \rightarrow \mathcal{G}_{k+1}^{*,p}$ such that $g(A)^* A \in N$, for all $A \in \mathcal{O}(A_1)$. This way, the map $\mathcal{A}_k^{*,p}(A_0) \ni A \mapsto (g(A)^*, g(A)) \in N \times \mathcal{G}_{k+1}^{*,p}$ would provide a local diffeomorphism. To prove this, we work in local coordinates near A_1 , that is, we express the connections $A \in \mathcal{A}_k^{*,p}(A_0)$ near A_1 as $A = T_{g_0}(u)^*(A_0 + \alpha_0 + \alpha)$, for some $(u, \alpha) \in L_{k+1-1/p}^{*,p}(\text{Ad}(P|_{\partial B^4})) \times L_{0T,k}^p(T^*B^4 \otimes \text{Ad}(P))$, with $\|u\|_{k+1-1/p,p}$ and $\|\alpha\|_{k,p}$ small, and prove the following lemma.

Lemma 2.3 *For $(u, \alpha) \in L_{k+1-1/p}^{*,p}(\text{Ad}(P|_{\partial B^4})) \times L_{0T,k}^p(T^*B^4 \otimes \text{Ad}(P))$, with $\|u\|_{k+1-1/p,p}$ and $\|\alpha\|_{k,p}$ suitably small, there exists $g(u, \alpha) \in \mathcal{G}_{k+1}^{*,p}$ such that $g(u, \alpha)^* A \in N$. Moreover, $g(u, \alpha)$ depends smoothly on u and α , and $g(0, 0) = \mathbf{1}$.*

Proof: Since we look for g near the identity, we may assume that g is of the form $g = \text{Exp}_1 \xi$ for $\xi \in L_{k+1}^{*,p}(\text{Ad}(P))$, with $\|\xi\|_{k+1,p}$ small. There exists $\epsilon > 0$ such that the map $f : \{u \in L_{k+1-1/p}^{*,p}(\text{Ad}(P|_{\partial B^4})) : \|u\|_{k+1-1/p,p} < \epsilon\} \times \{\xi \in L_{k+1}^{*,p}(\text{Ad}(P|_{\partial B^4})) : \|\xi\|_{k+1,p} < \epsilon\} \rightarrow L_{k+1-1/p}^{*,p}(\text{Ad}(P|_{\partial B^4}))$, defined via $\text{Exp}_1 u \cdot \text{Exp}_1 \xi = \text{Exp}_1(f(u, \xi))$, is smooth.

One has

$$\begin{aligned} g^* A &= T_{g_0}(f(u, \xi))^*(A_0 + \alpha_0 + \alpha) + (T_{g_0}(u)g)^*(A_0 + \alpha_0 + \alpha) - T_{g_0}(f(u, \xi))^*(A_0 + \alpha_0 + \alpha) \\ &= T_{g_0}(f(u, \xi))^*(A_0 + \alpha_0 + \alpha + \beta), \end{aligned}$$

where $\beta = \beta(u, \xi) = T_{g_0}(f(u, \xi))[(T_{g_0}(u)g)^*(A_0 + \alpha_0 + \alpha) - T_{g_0}(f(u, \xi))^*(A_0 + \alpha_0 + \alpha)]T_{g_0}(f(u, \xi))^{-1}$.

Hence,

$$\Phi_{g_0}^{-1}(g^* A) = (f(u, \xi), \alpha_0 + \alpha + \beta(u, \xi)).$$

Thus, we must find $\xi \in L_{k+1}^{*,p}(\text{Ad}(P))$ which satisfies the following equations for small $\|u\|_{k+1-1/p,p}$ and $\|\alpha\|_{k,p}$:

$$f(u, \xi) = 0 \quad \text{on } \partial B^4, \quad (2.5)$$

$$d_{A_0 + \alpha_0}^*(\alpha + \beta(u, \xi)) = 0 \quad \text{in } B^4. \quad (2.6)$$

The implicit function theorem yields the existence of a solution for the system (2.5)-(2.6) as follows: define

$$F : L_{k+1-1/p}^{*,p}(\text{Ad}(P|_{\partial B^4})) \times L_{0T,k}^p(T^* B^4 \otimes \text{Ad}(P)) \times L_{k+1}^{*,p}(\text{Ad}(P)) \rightarrow L_{k+1-1/p}^{*,p}(\text{Ad}(P|_{\partial B^4})) \times L_{k-1}^p(\text{Ad}(P))$$

by

$$F(u, \alpha, \xi) := (f(u, \xi)|_{\partial B^4}, d_{A_0 + \alpha_0}^*(\alpha + \beta(u, \xi))),$$

where $g = \text{Exp}_1 \xi$. For $\eta \in L_{k+1}^{*,p}(\text{Ad}(P))$, a simple calculation yields

$$\left\langle \frac{\partial F}{\partial \xi}(0, 0, 0), \eta \right\rangle = \frac{d}{dt} \Big|_{t=0} F(0, 0, t\eta) = (\eta|_{\partial B^4}, d_{A_0 + \alpha_0}^* d_{A_0 + \alpha_0}(g_0(\eta - T_0(\eta))g_0^{-1})).$$

We claim that $\frac{\partial F}{\partial \xi}(0, 0, 0) : L_{k+1}^{*,p}(\text{Ad}(P)) \rightarrow L_{k+1-1/p}^{*,p}(\text{Ad}(P|_{\partial B^4})) \times L_{k-1}^p(\text{Ad}(P))$ is an isomorphism. To see this, let $\eta \in \ker \frac{\partial F}{\partial \xi}(0, 0, 0)$. Then $\eta|_{\partial B^4} = 0$ and, also, $d_{A_0 + \alpha_0}^* d_{A_0 + \alpha_0}(g_0(\eta - T_0(\eta))g_0^{-1}) = 0$. These entail $T_0(\eta) = 0$ and $\eta = 0$. Thus, $\frac{\partial F}{\partial \xi}(0, 0, 0)$ is one-to-one. Furthermore, $\frac{\partial F}{\partial \xi}(0, 0, 0)$ is onto, since:

(i) For all $\varphi \in L_{0,k+1}^p(\text{Ad}(P))$ and $\eta \in L_{k+1}^{*,p}(\text{Ad}(P))$, we have $T_0((\eta + \varphi)|_{\partial B^4}) = T_0(\eta|_{\partial B^4})$. (Thus, $(\varphi + \eta) - T_0((\varphi + \eta)|_{\partial B^4}) = \varphi + \eta - T_0(\eta|_{\partial B^4})$).

(ii) $\text{Ad}(g_0^{-1}) : L_{0,k+1}^p(\text{Ad}(P)) \rightarrow L_{0,k+1}^p(\text{Ad}(P))$ is an isomorphism.

(iii) $\Delta_{A_0 + \alpha_0} := d_{A_0 + \alpha_0}^* d_{A_0 + \alpha_0} : L_{0,k+1}^p(\text{Ad}(P)) \rightarrow L_{k-1}^p(\text{Ad}(P))$ is an isomorphism.

Thus $\frac{\partial F}{\partial \xi}(0, 0, 0)$ is an isomorphism and, by the implicit function theorem, there exists a neighborhood \mathcal{U} of $(0, \alpha_0) \in U \times L_{0T,k}^p(T^* \otimes \text{Ad}(P))$ such that for all $(u, \alpha_0 + \alpha) \in \mathcal{U}$, there exists $g(u, \alpha) \in \mathcal{G}_{k+1}^{*,p}$ satisfying $g(u, \alpha)^* A \in N$, for $A = T_{g_0}(u)^*(A_0 + \alpha_0 + \alpha)$. This completes the proof. \square

The next lemma can be proved similarly to the corollary at p. 50 of [2], so we omit its proof.

Lemma 2.4 Assume $p(k+1) > 4$. Then $\mathcal{B}_k^{*,p}(A_0)$ is Hausdorff.

In order to complete the proof of Proposition 2.2, we also need the following:

Lemma 2.5 For any given $(0, \alpha_0) \in N$, there exists $\epsilon > 0$ such that the ϵ -ball of N centered at $(0, \alpha_0)$ injects into $\mathcal{B}_k^{*,p}(A_0)$.

Proof: Let $A_1 = g_0^*(A_0 + \alpha_0)$ (where $g_0 \in \mathcal{G}_{k+1}^{*,p}$, and $\alpha_0 \in L_{0T,k}^p(T^*B^4 \otimes \text{Ad}(P))$). Let $(0, \alpha_0 + \alpha), (0, \alpha_0 + \beta) \in N$, with $\|\alpha\|_{k,p} < \epsilon$ and $\|\beta\|_{k,p} < \epsilon$, where $\epsilon > 0$ is a small number. Set $A' = g_0^*(A_0 + \alpha_0 + \alpha)$, $A'' = g_0^*(A_0 + \alpha_0 + \beta)$. Assume that there exists $g \in \mathcal{G}_{k+1}^{*,p}$ such that $g^*A' = A''$. Under these assumptions, we need to show that $g = \mathbf{1}$ and $\alpha = \beta$. To prove this, we rewrite the condition $g^*A' = A''$ as

$$d_{A_1}g = g(g_0^{-1}\alpha g_0) - (g_0^{-1}\beta g_0)g. \quad (2.7)$$

Let us now write $g := \tilde{g} + c \in L_{k+1}^p(\text{End}(P))$, where $c \in \ker d_{A_1}$, $\tilde{g} \in (\ker d_{A_1})^\perp$ (here, $(\ker d_{A_1})^\perp$ is the L^2 -orthogonal complement of $\ker d_{A_1}$ in $L_{k+1}^p(\text{End}(P))$). By (2.7), there exists a constant $C > 0$ such that

$$\|\tilde{g}\|_{p,k+1} \leq C\epsilon. \quad (2.8)$$

On the other hand, since $g((0, 0, 0, 1)) = \mathbf{1}$, $c \simeq \mathbf{1}$ and, by (2.8), $\|g - \mathbf{1}\|_{k+1,p} < C\epsilon$. By Lemma 2.3, one has $g = \mathbf{1}$ and $\alpha = \beta$. This completes the proof. \square

3 Interaction estimates and construction of orthonormal bases for $T_{A(\mathbf{q})}\mathcal{N}(d_0, \lambda_0)$ and $T_{\tilde{A}(\mathbf{q})}\tilde{\mathcal{N}}(d_0, \lambda_0)$

In [4], for technical reasons, to the purpose of transforming the non compact ϵ -Dirichlet problem into a finite dimensional problem, we introduced the extensions $\tilde{P}(\mathbf{q})$ to \mathbb{R}^4 of the bundles $P(\mathbf{q})$,

$$\tilde{P}(\mathbf{q}) := \left(\{B_{\lambda/4}(p), \mathbb{R}^4 \setminus \{p\}\}, \{gg_{12,p}g^{-1}\} \right), \quad \text{for } \mathbf{q} \in \mathcal{P}(d_0, \lambda_0), \quad (3.1)$$

and the spaces of connections

$$\tilde{\mathcal{N}}(d_0, \lambda_0) := \{\tilde{A}(\mathbf{q}) : \mathbf{q} \in \mathcal{P}(d_0, \lambda_0)\}, \quad (3.2)$$

where

$$\tilde{A}(\mathbf{q}) = \begin{cases} \frac{1}{\epsilon}gI_{\lambda,p}^2g^{-1} & \text{in } \mathbb{R}^4 \setminus \{p\} \\ \frac{1}{\epsilon}gI_{\lambda,p}^1g^{-1} & \text{in } B_{\lambda/4}(p). \end{cases} \quad (3.3)$$

In this section, we prove some technical lemmas on the tangent space to approximate solutions $T_{A(\mathbf{q})}\mathcal{N}(d_0, \lambda_0)$ (cf. §1) and on the tangent space $T_{\tilde{A}(\mathbf{q})}\tilde{\mathcal{N}}(d_0, \lambda_0)$. These lemmas have been used in sections §3.5-§3.8 of [4], where, by a standard technique for non-compact variational problems, the Yang Mills equation is first solved in $T_{A(\mathbf{q})}\mathcal{N}(d_0, \lambda_0)^\perp$ (the orthogonal complement to the tangent space to approximate solutions), i.e., essentially, orthogonally to the kernel of the

Hessian of the ϵ -Yang Mills functional (cf., in particular, Lemma 3.9 in [4]). This result allowed us to turn the problem into a finite dimensional one (cf. Proposition 3.2 in [4]).

In the following, we choose (ξ_1, ξ_2, ξ_3) , where $\xi_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$, $\xi_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$, $\xi_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, as basis for the Lie algebra $\mathfrak{so}(3)$. Right translation by g yields a basis for $T_{[g]}SO(3)$, which we denote by $(\xi_1[g], \xi_2[g], \xi_3[g])$.

Lemma 3.1 *The following estimates hold for small $\epsilon > 0$:*

$$\left\| \frac{\partial A}{\partial p_i}(\mathbf{q}) \right\|_{A(\mathbf{q});1,2;B^4} \simeq \epsilon^{-3/2} \quad (1 \leq i \leq 4), \quad (3.4)$$

$$\left\| \frac{\partial A}{\partial \xi_i[g]}(\mathbf{q}) \right\|_{A(\mathbf{q});1,2;B^4} \simeq \epsilon^{-1} \quad (1 \leq i \leq 3), \quad (3.5)$$

$$\left\| \frac{\partial A}{\partial \lambda}(\mathbf{q}) \right\|_{A(\mathbf{q});1,2;B^4} \simeq \epsilon^{-3/2}, \quad (3.6)$$

$$\left| \left(\frac{\partial A}{\partial p_i}(\mathbf{q}), \frac{\partial A}{\partial p_j}(\mathbf{q}) \right)_{A(\mathbf{q});1,2;B^4} \right| \lesssim \epsilon^{-3/2} \quad (1 \leq i \neq j \leq 4), \quad (3.7)$$

$$\left| \left(\frac{\partial A}{\partial \xi_i[g]}(\mathbf{q}), \frac{\partial A}{\partial \xi_j[g]}(\mathbf{q}) \right)_{A(\mathbf{q});1,2;B^4} \right| \lesssim \epsilon^{-1} \quad (1 \leq i \neq j \leq 3), \quad (3.8)$$

$$\left| \left(\frac{\partial A}{\partial p_i}(\mathbf{q}), \frac{\partial A}{\partial \xi_j[g]}(\mathbf{q}) \right)_{A(\mathbf{q});1,2;B^4} \right| \lesssim \epsilon^{-1} \quad (1 \leq i \leq 4, 1 \leq j \leq 3), \quad (3.9)$$

$$\left| \left(\frac{\partial A}{\partial p_i}(\mathbf{q}), \frac{\partial A}{\partial \lambda}(\mathbf{q}) \right)_{A(\mathbf{q});1,2;B^4} \right| \lesssim \epsilon^{-2} \quad (1 \leq i \leq 4), \quad (3.10)$$

$$\left| \left(\frac{\partial A}{\partial \xi_i[g]}(\mathbf{q}), \frac{\partial A}{\partial \lambda}(\mathbf{q}) \right)_{A(\mathbf{q});1,2;B^4} \right| \lesssim \epsilon^{-3/2} \quad (1 \leq i \leq 3). \quad (3.11)$$

Proof: For $A(\mathbf{q})$ in the same gauge as described in §1, one has (modulo the action of infinitesimal gauge transformations of $P(\mathbf{q})$):

$$\begin{aligned} \frac{\partial A}{\partial p_i}(\mathbf{q}) = & -\frac{\partial \beta_{\lambda,p}}{\partial p_i} \underline{A}_\epsilon + \frac{1}{\epsilon} \frac{\partial \beta_{\lambda/4,p}}{\partial p_i} g I_{\lambda,p}^2 g^{-1} + \frac{1}{\epsilon} \beta_{\lambda/4,p} g \frac{\partial I_{\lambda,p}^2}{\partial p_i} g^{-1} + \\ & -\frac{1}{\epsilon} \frac{\partial \beta_{\lambda/4,p}}{\partial p_i} g P I_{\lambda,p}^2 g^{-1} + \frac{1}{\epsilon} (1 - \beta_{\lambda/4,p}) g \frac{\partial P I_{\lambda,p}^2}{\partial p_i} g^{-1} \quad \text{in } B^4 \setminus \{p\}, \end{aligned} \quad (3.12)$$

$$\frac{\partial A}{\partial p_i}(\mathbf{q}) = \frac{1}{\epsilon} g \frac{\partial I_{\lambda,p}^1}{\partial p_i} g^{-1} \quad \text{in } B_{\lambda/4}(p), \quad (3.13)$$

$$\frac{\partial A}{\partial \xi_i[g]}(\mathbf{q}) = \frac{1}{\epsilon} \beta_{\lambda/4,p} [\xi_i, g I_{\lambda,p}^2 g^{-1}] + \frac{1}{\epsilon} (1 - \beta_{\lambda/4,p}) [\xi_i, g P I_{\lambda,p}^2 g^{-1}] \quad \text{in } B^4 \setminus \{p\}, \quad (3.14)$$

$$\frac{\partial A}{\partial \xi_i[g]}(\mathbf{q}) = \frac{1}{\epsilon} [\xi_i, g I_{\lambda,p}^1 g^{-1}] \quad \text{in } B_{\lambda/4}(p), \quad (3.15)$$

$$\begin{aligned} \frac{\partial A}{\partial \lambda}(\mathbf{q}) = & -\frac{\partial \beta_{\lambda,p}}{\partial \lambda} \underline{A}_\epsilon + \frac{1}{\epsilon} \frac{\partial \beta_{\lambda/4,p}}{\partial \lambda} g I_{\lambda,p}^2 g^{-1} + \frac{1}{\epsilon} \beta_{\lambda/4,p} g \frac{\partial I_{\lambda,p}^2}{\partial \lambda} g^{-1} + \\ & -\frac{1}{\epsilon} \frac{\partial \beta_{\lambda/4,p}}{\partial \lambda} g P I_{\lambda,p}^2 g^{-1} + \frac{1}{\epsilon} (1 - \beta_{\lambda/4,p}) g \frac{\partial P I_{\lambda,p}^2}{\partial \lambda} g^{-1} \quad \text{in } B^4 \setminus \{p\}, \end{aligned} \quad (3.16)$$

$$\frac{\partial A}{\partial \lambda}(\mathbf{q}) = \frac{1}{\epsilon} g \frac{\partial I_{\lambda,p}^1}{\partial \lambda} g^{-1} \quad \text{in } B_{\lambda/4}(p). \quad (3.17)$$

The result follows from these formulas by direct calculation. \square

From Lemma 3.1, it follows that $T_{A(\mathbf{q})}\mathcal{N}(d_0, \lambda_0)$ is an 8-dimensional linear space, if $\epsilon > 0$ is suitably small.

Let us recall that, for a given connection A on the bundle $P(\mathbf{q})$, the inner product $(\alpha, \beta)_{A;1,2;B^4}$ for one-forms α, β in $L^2_{1;A}(T^*B^4 \otimes \text{Ad}(P))$ is defined as:

$$(\alpha, \beta)_{A;1,2;B^4} := \int_{B^4} (\nabla_A^\epsilon \alpha, \nabla_A^\epsilon \beta) dx + \int_{B^4} (\alpha, \beta) dx \quad (3.18)$$

with $\nabla_A^\epsilon = \nabla + \epsilon[A, \cdot]$.

As a corollary of Lemma 3.1, one obtains the following orthonormal basis for $T_{A(\mathbf{q})}\mathcal{N}(d_0, \lambda_0)$, the tangent space to approximate solutions.

Lemma 3.2 *There exists a basis $\langle \mathbf{a}_1(\mathbf{q}), \mathbf{a}_2(\mathbf{q}), \dots, \mathbf{a}_8(\mathbf{q}) \rangle$ for $T_{A(\mathbf{q})}\mathcal{N}(d_0, \lambda_0)$, orthonormal with respect to the inner product $(\cdot, \cdot)_{A(\mathbf{q});1,2;B^4}$, of the following form:*

$$\mathbf{a}_1(\mathbf{q}) = a_{11}(\mathbf{q}) \frac{\partial A}{\partial p_1}(\mathbf{q}), \quad \text{with } a_{11}(\mathbf{q}) \simeq \epsilon^{3/2}, \quad (3.19)$$

$$\mathbf{a}_2(\mathbf{q}) = a_{22}(\mathbf{q}) \frac{\partial A}{\partial p_2} + a_{21}(\mathbf{q}) \mathbf{a}_1(\mathbf{q}), \quad \text{with } a_{22}(\mathbf{q}) \simeq \epsilon^{3/2}, |a_{21}(\mathbf{q})| \lesssim \epsilon^{3/2}, \quad (3.20)$$

$$\begin{aligned} \mathbf{a}_3(\mathbf{q}) &= a_{33}(\mathbf{q}) \frac{\partial A}{\partial p_3}(\mathbf{q}) + a_{32}(\mathbf{q}) \mathbf{a}_2(\mathbf{q}) + a_{31}(\mathbf{q}) \mathbf{a}_1(\mathbf{q}), \\ \text{with } a_{33}(\mathbf{q}) &\simeq \epsilon^{3/2}, |a_{3i}(\mathbf{q})| \lesssim \epsilon^{3/2} \quad (i = 1, 2), \end{aligned} \quad (3.21)$$

$$\begin{aligned} \mathbf{a}_4(\mathbf{q}) &= a_{44}(\mathbf{q}) \frac{\partial A}{\partial p_4}(\mathbf{q}) + a_{43}(\mathbf{q}) \mathbf{a}_3(\mathbf{q}) + a_{42}(\mathbf{q}) \mathbf{a}_2(\mathbf{q}) + a_{41}(\mathbf{q}) \mathbf{a}_1(\mathbf{q}), \\ \text{with } a_{44}(\mathbf{q}) &\simeq \epsilon^{3/2}, |a_{4i}(\mathbf{q})| \lesssim \epsilon^{3/2} \quad (1 \leq i \leq 3), \end{aligned} \quad (3.22)$$

$$\begin{aligned} \mathbf{a}_5(\mathbf{q}) &= a_{55}(\mathbf{q}) \frac{\partial A}{\partial \xi_1[g]}(\mathbf{q}) + a_{54}(\mathbf{q}) \mathbf{a}_4(\mathbf{q}) + a_{53}(\mathbf{q}) \mathbf{a}_3(\mathbf{q}) + a_{52}(\mathbf{q}) \mathbf{a}_2(\mathbf{q}) + a_{51}(\mathbf{q}) \mathbf{a}_1(\mathbf{q}), \\ \text{with } a_{55}(\mathbf{q}) &\simeq \epsilon, |a_{5i}(\mathbf{q})| \lesssim \epsilon^{3/2} \quad (1 \leq i \leq 4), \end{aligned} \quad (3.23)$$

$$\begin{aligned} \mathbf{a}_6(\mathbf{q}) &= a_{66}(\mathbf{q}) \frac{\partial A}{\partial \xi_2[g]}(\mathbf{q}) + a_{65}(\mathbf{q}) \mathbf{a}_5(\mathbf{q}) + a_{64}(\mathbf{q}) \mathbf{a}_4(\mathbf{q}) + a_{63}(\mathbf{q}) \mathbf{a}_3(\mathbf{q}) \\ &+ a_{62}(\mathbf{q}) \mathbf{a}_2(\mathbf{q}) + a_{61}(\mathbf{q}) \mathbf{a}_1(\mathbf{q}), \\ \text{with } a_{66}(\mathbf{q}) &\simeq \epsilon, |a_{65}(\mathbf{q})| \lesssim \epsilon, |a_{6i}(\mathbf{q})| \lesssim \epsilon^{3/2} \quad (1 \leq i \leq 4), \end{aligned} \quad (3.24)$$

$$\begin{aligned} \mathbf{a}_7(\mathbf{q}) &= a_{77}(\mathbf{q}) \frac{\partial A}{\partial \xi_3[g]}(\mathbf{q}) + a_{76}(\mathbf{q}) \mathbf{a}_6(\mathbf{q}) + a_{75}(\mathbf{q}) \mathbf{a}_5(\mathbf{q}) + a_{74}(\mathbf{q}) \mathbf{a}_4(\mathbf{q}) \\ &+ a_{73}(\mathbf{q}) \mathbf{a}_3(\mathbf{q}) + a_{72}(\mathbf{q}) \mathbf{a}_2(\mathbf{q}) + a_{71}(\mathbf{q}) \mathbf{a}_1(\mathbf{q}), \\ \text{with } a_{77}(\mathbf{q}) &\simeq \epsilon, |a_{76}(\mathbf{q})|, |a_{75}(\mathbf{q})| \lesssim \epsilon, |a_{7i}(\mathbf{q})| \lesssim \epsilon^{3/2} \quad (1 \leq i \leq 4), \end{aligned} \quad (3.25)$$

$$\begin{aligned} \mathbf{a}_8(\mathbf{q}) &= a_{88}(\mathbf{q}) \frac{\partial A}{\partial \lambda}(\mathbf{q}) + a_{87}(\mathbf{q}) \mathbf{a}_7(\mathbf{q}) + a_{86}(\mathbf{q}) \mathbf{a}_6(\mathbf{q}) + a_{85}(\mathbf{q}) \mathbf{a}_5(\mathbf{q}) + a_{84}(\mathbf{q}) \mathbf{a}_4(\mathbf{q}) \\ &+ a_{83}(\mathbf{q}) \mathbf{a}_3(\mathbf{q}) + a_{82}(\mathbf{q}) \mathbf{a}_2(\mathbf{q}) + a_{81}(\mathbf{q}) \mathbf{a}_1(\mathbf{q}), \\ \text{with } a_{88}(\mathbf{q}) &\simeq \epsilon^{3/2}, |a_{8i}(\mathbf{q})| \lesssim \epsilon \quad (1 \leq i \leq 7). \end{aligned} \quad (3.26)$$

Proof: We apply the Gram-Schmidt's orthogonalization procedure to construct $\mathbf{a}_i(\mathbf{q})$ ($1 \leq i \leq 8$) from $\frac{\partial A}{\partial p_i}(\mathbf{q})$ ($1 \leq i \leq 4$), $\frac{\partial A}{\partial \xi_i[g]}(\mathbf{q})$ ($1 \leq i \leq 3$), and $\frac{\partial A}{\partial \lambda}(\mathbf{q})$. The asserted result follows from Lemma 3.1 by direct calculation. \square

We construct the following vector fields $\mathbf{q}_i = \mathbf{q}_i(\mathbf{q})$ ($1 \leq i \leq 8$) inductively on $\mathcal{P}(d_0, \lambda_0)$, in such a way that the derivative of $A(\mathbf{q})$ in the direction $\mathbf{q}_i(\mathbf{q})$ yields $\mathbf{a}_i(\mathbf{q})$, i.e., $\mathbf{a}_i(\mathbf{q}) = A_{\mathbf{q}_i}(\mathbf{q})$ for $1 \leq i \leq 8$. More precisely,

$$\mathbf{q}_1(\mathbf{q}) = a_{11}(\mathbf{q}) \frac{\partial}{\partial p_1}, \quad (3.27)$$

$$\mathbf{q}_2(\mathbf{q}) = a_{22}(\mathbf{q}) \frac{\partial}{\partial p_2} + a_{21}(\mathbf{q}) \mathbf{q}_1(\mathbf{q}), \quad (3.28)$$

$$\mathbf{q}_3(\mathbf{q}) = a_{33}(\mathbf{q}) \frac{\partial}{\partial p_3} + a_{32}(\mathbf{q}) \mathbf{q}_2(\mathbf{q}) + a_{31}(\mathbf{q}) \mathbf{q}_1(\mathbf{q}), \quad (3.29)$$

$$\mathbf{q}_4(\mathbf{q}) = a_{44}(\mathbf{q}) \frac{\partial}{\partial p_4} + a_{43}(\mathbf{q}) \mathbf{q}_3(\mathbf{q}) + a_{42}(\mathbf{q}) \mathbf{q}_2(\mathbf{q}) + a_{41}(\mathbf{q}) \mathbf{q}_1(\mathbf{q}), \quad (3.30)$$

$$\mathbf{q}_5(\mathbf{q}) = a_{55}(\mathbf{q}) \xi_1[g] + a_{54}(\mathbf{q}) \mathbf{q}_4(\mathbf{q}) + a_{53}(\mathbf{q}) \mathbf{q}_3(\mathbf{q}) + a_{52}(\mathbf{q}) \mathbf{q}_2(\mathbf{q}) + a_{51}(\mathbf{q}) \mathbf{q}_1(\mathbf{q}), \quad (3.31)$$

$$\begin{aligned} \mathbf{q}_6(\mathbf{q}) = & a_{66}(\mathbf{q}) \xi_2[g] + a_{65}(\mathbf{q}) \mathbf{q}_5(\mathbf{q}) + a_{64}(\mathbf{q}) \mathbf{q}_4(\mathbf{q}) + a_{63}(\mathbf{q}) \mathbf{q}_3(\mathbf{q}) \\ & + a_{62}(\mathbf{q}) \mathbf{q}_2(\mathbf{q}) + a_{61}(\mathbf{q}) \mathbf{q}_1(\mathbf{q}), \end{aligned} \quad (3.32)$$

$$\begin{aligned} \mathbf{q}_7(\mathbf{q}) = & a_{77}(\mathbf{q}) \xi_3[g] + a_{76}(\mathbf{q}) \mathbf{q}_6(\mathbf{q}) + a_{75}(\mathbf{q}) \mathbf{q}_5(\mathbf{q}) + a_{74}(\mathbf{q}) \mathbf{q}_4(\mathbf{q}) + a_{73}(\mathbf{q}) \mathbf{q}_3(\mathbf{q}) \\ & + a_{72}(\mathbf{q}) \mathbf{q}_2(\mathbf{q}) + a_{71}(\mathbf{q}) \mathbf{q}_1(\mathbf{q}), \end{aligned} \quad (3.33)$$

$$\begin{aligned} \mathbf{q}_8(\mathbf{q}) = & a_{88}(\mathbf{q}) \frac{\partial}{\partial \lambda} + a_{87}(\mathbf{q}) \mathbf{q}_7(\mathbf{q}) + a_{86}(\mathbf{q}) \mathbf{q}_6(\mathbf{q}) + a_{85}(\mathbf{q}) \mathbf{q}_5(\mathbf{q}) + a_{84}(\mathbf{q}) \mathbf{q}_4(\mathbf{q}) \\ & + a_{83}(\mathbf{q}) \mathbf{q}_3(\mathbf{q}) + a_{82}(\mathbf{q}) \mathbf{q}_2(\mathbf{q}) + a_{81}(\mathbf{q}) \mathbf{q}_1(\mathbf{q}). \end{aligned} \quad (3.34)$$

Let now $\tilde{\mathbf{a}}_i(\mathbf{q}) = \tilde{A}_{\mathbf{q}_i}(\mathbf{q})$ (for $1 \leq i \leq 8$), where $\tilde{A}(\mathbf{q})$ are the instanton solutions introduced at the beginning of this section. In the following lemma, we construct a special basis for $T_{\tilde{A}(\mathbf{q})} \tilde{N}(d_0, \lambda_0)$, orthonormal with respect to the weighted inner product on $L^2_{1, \tilde{A}(\mathbf{q})}(T^* \mathbb{R}^4 \otimes \text{Ad}(\tilde{P}))$ defined by

$$(\alpha, \beta)_{1,2; \tilde{A}(\mathbf{q})} := \int_{\mathbb{R}^4} (\nabla_{\tilde{A}(\mathbf{q})}^\epsilon \alpha, \nabla_{\tilde{A}(\mathbf{q})}^\epsilon \beta) dx + \int_{\mathbb{R}^4} w(x) (\alpha, \beta) dx, \quad (3.35)$$

where $w(x) = 1$ for $|x| \leq 1$, and $w(x) = 1/(1 + |x|^2)^2$ for $|x| > 1$.

Lemma 3.3 *There exists a basis $\langle \hat{\mathbf{a}}_1(\mathbf{q}), \hat{\mathbf{a}}_2(\mathbf{q}), \dots, \hat{\mathbf{a}}_8(\mathbf{q}) \rangle$ for $T_{\tilde{A}(\mathbf{q})} \tilde{N}(d_0, \lambda_0)$, orthonormal with*

respect to the inner product $(\cdot, \cdot)_{\tilde{A}(\mathbf{q});1,2;\mathbb{R}^4}$, of the following form:

$$\hat{\mathbf{a}}_1(\mathbf{q}) = b_{11}(\mathbf{q})\tilde{\mathbf{a}}_1(\mathbf{q}), \quad \text{with } |b_{11}(\mathbf{q}) - 1| \lesssim \epsilon, \quad (3.36)$$

$$\hat{\mathbf{a}}_2(\mathbf{q}) = b_{22}(\mathbf{q})\tilde{\mathbf{a}}_2(\mathbf{q}) + b_{21}(\mathbf{q})\hat{\mathbf{a}}_1(\mathbf{q}), \quad \text{with } |b_{22}(\mathbf{q}) - 1| \lesssim \epsilon, \quad |b_{21}(\mathbf{q})| \lesssim \epsilon, \quad (3.37)$$

$$\begin{aligned} \hat{\mathbf{a}}_3(\mathbf{q}) &= b_{33}(\mathbf{q})\tilde{\mathbf{a}}_3(\mathbf{q}) + b_{32}(\mathbf{q})\hat{\mathbf{a}}_2(\mathbf{q}) + b_{31}(\mathbf{q})\hat{\mathbf{a}}_1(\mathbf{q}), \\ &\text{with } |b_{33}(\mathbf{q}) - 1| \lesssim \epsilon, \quad |b_{3i}(\mathbf{q})| \lesssim \epsilon \quad (i = 1, 2), \end{aligned} \quad (3.38)$$

$$\begin{aligned} \hat{\mathbf{a}}_4(\mathbf{q}) &= b_{44}(\mathbf{q})\tilde{\mathbf{a}}_4(\mathbf{q}) + b_{43}(\mathbf{q})\hat{\mathbf{a}}_3(\mathbf{q}) + b_{42}(\mathbf{q})\hat{\mathbf{a}}_2(\mathbf{q}) + b_{41}(\mathbf{q})\hat{\mathbf{a}}_1(\mathbf{q}), \\ &\text{with } |b_{44}(\mathbf{q}) - 1| \lesssim \epsilon, \quad |b_{4i}(\mathbf{q})| \lesssim \epsilon \quad (1 \leq i \leq 3), \end{aligned} \quad (3.39)$$

$$\begin{aligned} \hat{\mathbf{a}}_5(\mathbf{q}) &= b_{55}(\mathbf{q})\tilde{\mathbf{a}}_5(\mathbf{q}) + b_{54}(\mathbf{q})\hat{\mathbf{a}}_4(\mathbf{q}) + b_{53}(\mathbf{q})\hat{\mathbf{a}}_3(\mathbf{q}) + b_{52}(\mathbf{q})\hat{\mathbf{a}}_2(\mathbf{q}) + b_{51}(\mathbf{q})\hat{\mathbf{a}}_1(\mathbf{q}), \\ &\text{with } |b_{55}(\mathbf{q}) - 1| \lesssim \epsilon, \quad |b_{5i}(\mathbf{q})| \lesssim \epsilon \quad (1 \leq i \leq 4), \end{aligned} \quad (3.40)$$

$$\begin{aligned} \hat{\mathbf{a}}_6(\mathbf{q}) &= b_{66}(\mathbf{q})\tilde{\mathbf{a}}_6(\mathbf{q}) + b_{65}(\mathbf{q})\hat{\mathbf{a}}_5(\mathbf{q}) + b_{64}(\mathbf{q})\hat{\mathbf{a}}_4(\mathbf{q}) + b_{63}(\mathbf{q})\hat{\mathbf{a}}_3(\mathbf{q}) \\ &\quad + b_{62}(\mathbf{q})\hat{\mathbf{a}}_2(\mathbf{q}) + b_{61}(\mathbf{q})\hat{\mathbf{a}}_1(\mathbf{q}), \quad \text{with } |b_{66}(\mathbf{q}) - 1| \lesssim \epsilon, \quad |b_{6i}(\mathbf{q})| \lesssim \epsilon \quad (1 \leq i \leq 5), \end{aligned} \quad (3.41)$$

$$\begin{aligned} \hat{\mathbf{a}}_7(\mathbf{q}) &= b_{77}(\mathbf{q})\tilde{\mathbf{a}}_7(\mathbf{q}) + b_{76}(\mathbf{q})\hat{\mathbf{a}}_6(\mathbf{q}) + b_{75}(\mathbf{q})\hat{\mathbf{a}}_5(\mathbf{q}) + b_{74}(\mathbf{q})\hat{\mathbf{a}}_4(\mathbf{q}) + b_{73}(\mathbf{q})\hat{\mathbf{a}}_3(\mathbf{q}) \\ &\quad + b_{72}(\mathbf{q})\hat{\mathbf{a}}_2(\mathbf{q}) + b_{71}(\mathbf{q})\hat{\mathbf{a}}_1(\mathbf{q}), \quad \text{with } |b_{77}(\mathbf{q}) - 1| \lesssim \epsilon, \quad |b_{7i}(\mathbf{q})| \lesssim \epsilon \quad (1 \leq i \leq 6), \end{aligned} \quad (3.42)$$

$$\begin{aligned} \hat{\mathbf{a}}_8(\mathbf{q}) &= b_{88}(\mathbf{q})\tilde{\mathbf{a}}_8(\mathbf{q}) + b_{87}(\mathbf{q})\hat{\mathbf{a}}_7(\mathbf{q}) + b_{86}(\mathbf{q})\hat{\mathbf{a}}_6(\mathbf{q}) + b_{85}(\mathbf{q})\hat{\mathbf{a}}_5(\mathbf{q}) + b_{84}(\mathbf{q})\hat{\mathbf{a}}_4(\mathbf{q}) \\ &\quad + b_{83}(\mathbf{q})\hat{\mathbf{a}}_3(\mathbf{q}) + b_{82}(\mathbf{q})\hat{\mathbf{a}}_2(\mathbf{q}) + b_{81}(\mathbf{q})\hat{\mathbf{a}}_1(\mathbf{q}), \quad \text{with } |b_{88}(\mathbf{q}) - 1| \lesssim \epsilon, \quad |b_{8i}(\mathbf{q})| \lesssim \epsilon \quad (1 \leq i \leq 7). \end{aligned} \quad (3.43)$$

Proof: We apply the Gram-Schmidt's orthogonalization procedure to construct $\hat{\mathbf{a}}_i(\mathbf{q})$ ($1 \leq i \leq 8$) from $\tilde{\mathbf{a}}_i(\mathbf{q})$ ($1 \leq i \leq 8$). In the gauge used for $\tilde{A}(\mathbf{q})$ (cf. beginning of §3), the derivatives of $\tilde{A}(\mathbf{q})$ are written as (modulo the action of the infinitesimal gauge transformation of $\tilde{P}(\mathbf{q})$):

$$\frac{\partial \tilde{A}}{\partial p_i}(\mathbf{q}) = \frac{1}{\epsilon} g \frac{\partial I_{\lambda,p}^2}{\partial p_i} g^{-1} \quad \text{in } \mathbb{R}^4 \setminus \{p\}, \quad \frac{\partial \tilde{A}}{\partial p_i}(\mathbf{q}) = \frac{1}{\epsilon} g \frac{\partial I_{\lambda,p}^1}{\partial p_i} g^{-1} \quad \text{in } B_{\lambda/4}(p), \quad (3.44)$$

$$\frac{\partial \tilde{A}}{\partial \xi_i[g]}(\mathbf{q}) = \frac{1}{\epsilon} [\xi_i, g I_{\lambda,p}^2 g^{-1}] \quad \text{in } \mathbb{R}^4 \setminus \{p\}, \quad \frac{\partial \tilde{A}}{\partial \xi_i[g]}(\mathbf{q}) = \frac{1}{\epsilon} [\xi_i, g I_{\lambda,p}^1 g^{-1}] \quad \text{in } B_{\lambda/4}(p), \quad (3.45)$$

$$\frac{\partial \tilde{A}}{\partial \lambda}(\mathbf{q}) = \frac{1}{\epsilon} \frac{\partial \tilde{A}}{\partial \lambda}(\mathbf{q}) \quad \text{in } \mathbb{R}^4 \setminus \{p\}, \quad \frac{\partial \tilde{A}}{\partial \lambda}(\mathbf{q}) = \frac{1}{\epsilon} \frac{\partial \tilde{A}}{\partial \lambda}(\mathbf{q}) \quad \text{in } B_{\lambda/4}(p) \quad (3.46)$$

From (3.44)-(3.46), by direct computation, one has

$$\|\tilde{\mathbf{a}}_i(\mathbf{q})\|_{\tilde{A}(\mathbf{q});1,2;\mathbb{R}^4 \setminus B^4} \lesssim \epsilon^{3/2} \quad (1 \leq i \leq 4), \quad \|\tilde{\mathbf{a}}_i(\mathbf{q})\|_{\tilde{A}(\mathbf{q});1,2;\mathbb{R}^4 \setminus B^4} \lesssim \epsilon \quad (5 \leq i \leq 8). \quad (3.47)$$

On the other hand,

$$(\tilde{\mathbf{a}}_i(\mathbf{q}), \tilde{\mathbf{a}}_j(\mathbf{q}))_{\tilde{A}(\mathbf{q});1,2} = \delta_{ij} + O(\epsilon), \quad (3.48)$$

for $1 \leq i, j \leq 8$. In fact,

$$\begin{aligned} (\tilde{\mathbf{a}}_i(\mathbf{q}), \tilde{\mathbf{a}}_j(\mathbf{q}))_{\tilde{A}(\mathbf{q});1,2} &= (\tilde{\mathbf{a}}_i(\mathbf{q}), \tilde{\mathbf{a}}_j(\mathbf{q}))_{\tilde{A}(\mathbf{q});1,2;B^4} + (\tilde{\mathbf{a}}_i(\mathbf{q}), \tilde{\mathbf{a}}_j(\mathbf{q}))_{\tilde{A}(\mathbf{q});1,2;\mathbb{R}^4 \setminus B^4} \\ &= (\tilde{\mathbf{a}}_i(\mathbf{q}), \tilde{\mathbf{a}}_j(\mathbf{q}))_{\tilde{A}(\mathbf{q});1,2;B^4} + O(\epsilon^2) \quad (\text{by (3.47)}) \\ &= (\mathbf{a}_i(\mathbf{q}) + \tilde{\mathbf{a}}_i(\mathbf{q}) - \mathbf{a}_i(\mathbf{q}), \mathbf{a}_j(\mathbf{q}) + \tilde{\mathbf{a}}_j(\mathbf{q}) - \mathbf{a}_j(\mathbf{q}))_{\tilde{A}(\mathbf{q});1,2;B^4} + O(\epsilon^2) \\ &= (\mathbf{a}_i(\mathbf{q}), \mathbf{a}_j(\mathbf{q}))_{\tilde{A}(\mathbf{q});1,2;B^4} + O(\epsilon) \quad (\text{by Lemma 3.4 below}). \end{aligned} \quad (3.49)$$

Here, by setting $b(\mathbf{q}) := \tilde{A}(\mathbf{q}) - A(\mathbf{q})$, one obtains

$$\begin{aligned}
(\mathbf{a}_i(\mathbf{q}), \mathbf{a}_j(\mathbf{q}))_{\tilde{A}(\mathbf{q});1,2;B^4} &= (\mathbf{a}_i(\mathbf{q}), \mathbf{a}_j(\mathbf{q}))_{A(\mathbf{q});1,2;B^4} + \int_{B^4} (\nabla_{\tilde{A}(\mathbf{q})}^\epsilon \mathbf{a}_i(\mathbf{q}), \nabla_{\tilde{A}(\mathbf{q})}^\epsilon \mathbf{a}_j(\mathbf{q})) dx \\
&\quad - \int_{B^4} (\nabla_{A(\mathbf{q})}^\epsilon \mathbf{a}_i(\mathbf{q}), \nabla_{A(\mathbf{q})}^\epsilon \mathbf{a}_j(\mathbf{q})) dx \\
&= \delta_{ij} + \epsilon \int_{B^4} (\nabla_{A(\mathbf{q})}^\epsilon \mathbf{a}_i(\mathbf{q}), [b(\mathbf{q}), \mathbf{a}_j(\mathbf{q})]) dx + \epsilon \int_{B^4} ([b(\mathbf{q}), \mathbf{a}_i(\mathbf{q})], \nabla_{A(\mathbf{q})}^\epsilon \mathbf{a}_j(\mathbf{q})) dx \\
&\quad + \epsilon^2 \int_{B^4} ([b(\mathbf{q}), \mathbf{a}_i(\mathbf{q})], [b(\mathbf{q}), \mathbf{a}_j(\mathbf{q})]) dx = \delta_{ij} + O(\epsilon), \tag{3.50}
\end{aligned}$$

since $\epsilon \|b(\mathbf{q})\|_\infty \lesssim \epsilon$.

Combining (3.49), (3.50), one obtains (3.48), hence the $\hat{\mathbf{a}}_i(\mathbf{q})$'s ($1 \leq i \leq 8$) satisfy the assertion of the lemma. \square

Lemma 3.4 For $\mathbf{q} \in \mathcal{P}(d_0, \lambda_0; D_1, D_2; \epsilon)$,

$$\begin{aligned}
\|\mathbf{a}_i(\mathbf{q}) - \tilde{\mathbf{a}}_i(\mathbf{q})\|_{A(\mathbf{q});1,2;B^4} &\lesssim \epsilon^{3/2} \quad (1 \leq i \leq 4), \\
\|\mathbf{a}_i(\mathbf{q}) - \tilde{\mathbf{a}}_i(\mathbf{q})\|_{A(\mathbf{q});1,2;B^4} &\lesssim \epsilon \quad (5 \leq i \leq 8).
\end{aligned}$$

Proof: The connections $A(\mathbf{q})$ and $\tilde{A}(\mathbf{q})$, represented in the same gauge as in the previous lemmas, differ only on the domain $B^4 \setminus B_{\lambda/4}(p)$. In particular, $A(\mathbf{q}) - \tilde{A}(\mathbf{q}) = (\beta_{\lambda,p}) \underline{A}_\epsilon + \frac{1}{\epsilon} (\beta_{\lambda/4,p} - 1) gh_{\lambda,p} g^{-1}$. One has (modulo infinitesimal gauge transformations):

$$\frac{\partial A}{\partial p_i}(\mathbf{q}) - \frac{\partial \tilde{A}}{\partial p_i}(\mathbf{q}) = -\frac{\partial \beta_{\lambda,p}}{\partial p_i} \underline{A}_\epsilon + \frac{1}{\epsilon} \frac{\partial \beta_{\lambda/4,p}}{\partial p_i} gh_{\lambda,p} g^{-1} + \frac{1}{\epsilon} (\beta_{\lambda/4,p} - 1) g \frac{\partial h_{\lambda,p}}{\partial p_i} g^{-1}, \tag{3.51}$$

$$\frac{\partial A}{\partial \xi_i[g]}(\mathbf{q}) - \frac{\partial \tilde{A}}{\partial \xi_i[g]}(\mathbf{q}) = \frac{1}{\epsilon} (\beta_{\lambda/4,p} - 1) [\xi_i, gh_{\lambda,p} g^{-1}], \tag{3.52}$$

$$\frac{\partial A}{\partial \lambda}(\mathbf{q}) - \frac{\partial \tilde{A}}{\partial \lambda}(\mathbf{q}) = -\frac{\partial \beta_{\lambda,p}}{\partial \lambda} \underline{A}_\epsilon + \frac{1}{\epsilon} \frac{\partial \beta_{\lambda/4,p}}{\partial \lambda} gh_{\lambda,p} g^{-1} + \frac{1}{\epsilon} (\beta_{\lambda/4,p} - 1) g \frac{\partial h_{\lambda,p}}{\partial \lambda} g^{-1}. \tag{3.53}$$

The asserted result follows from Lemma 3.2 by direct computation. \square

We recall that the Hessian of \mathcal{YM}_ϵ , denoted by $\nabla^2 \mathcal{YM}_\epsilon(A)$, is defined by

$$\langle \nabla^2 \mathcal{YM}_\epsilon(A) a, b \rangle := 2 \int_{B^4} (d_A^\epsilon a, d_A^\epsilon b) + 2 \int_{B^4} (F_A, \epsilon[a, b]) \quad \text{for all } a, b \in L_{1,0}^2(T^*B^4 \otimes \text{Ad}(P)), \tag{3.54}$$

where $\langle \cdot, \cdot \rangle$ denotes the pairing between $L_{1,0}^2(T^*B^4 \otimes \text{Ad}(P))$ and its dual and $d_A^\epsilon = d + \epsilon[A, \cdot]$.

Lemma 3.5 For $\mathbf{q} \in \mathcal{P}(d_0, \lambda_0; D_1, D_2; \epsilon)$,

$$\begin{aligned}
\|(\nabla^2 \mathcal{YM}_\epsilon(A(\mathbf{q})) - \nabla^2 \mathcal{YM}_\epsilon(\tilde{A}(\mathbf{q}))) \mathbf{a}_i(\mathbf{q})\|_{A(\mathbf{q});1,2,*} &\lesssim \epsilon^{3/2} \quad (1 \leq i \leq 8), \\
\|(d_{A(\mathbf{q})} d_{A(\mathbf{q})}^* - d_{\tilde{A}(\mathbf{q})} d_{\tilde{A}(\mathbf{q})}^*) \mathbf{a}_i(\mathbf{q})\|_{A(\mathbf{q});1,2,*} &\lesssim \epsilon^{3/2} \quad (1 \leq i \leq 8).
\end{aligned}$$

Proof: We perform the calculation for $i = 1$ (the remaining cases are analogous). For $b(\mathbf{q}) := \tilde{A}(\mathbf{q}) - A(\mathbf{q})$, $\alpha \in L_1^2(T^*B^4 \otimes \text{Ad}(P(\mathbf{q})))$, $\beta \in L_{0,1}^2(T^*B^4 \otimes \text{Ad}(P(\mathbf{q})))$, one has

$$\begin{aligned} & \frac{1}{2}(\nabla^2 \mathcal{M}_\epsilon(\tilde{A}(\mathbf{q})) - \nabla^2 \mathcal{M}_\epsilon(A(\mathbf{q})))(\alpha, \beta) \\ &= \epsilon \int_{B^4} (d_{A(\mathbf{q})}^\epsilon \alpha, [b(\mathbf{q}), \beta]) dx + \epsilon \int_{B^4} ([b(\mathbf{q}), \alpha], d_{A(\mathbf{q})}^\epsilon \beta) dx \\ & \quad + \epsilon \int_{B^4} (d_{A(\mathbf{q})}^\epsilon b(\mathbf{q}), [\alpha, \beta]) dx + \epsilon^2 \int_{B^4} ([b(\mathbf{q}), \alpha], [b(\mathbf{q}), \beta]) dx \\ & \quad + \frac{\epsilon^2}{2} \int_{B^4} ([b(\mathbf{q}), b(\mathbf{q})], [\alpha, \beta]) dx. \end{aligned} \quad (3.55)$$

We set $\alpha = \mathbf{a}_1(\mathbf{q})$ in (3.55), and estimate each term. In the following, we write $r := |x - p|$. Since

$$|\mathbf{a}_1(\mathbf{q})| = a_{11}(\mathbf{q}) \frac{\partial A}{\partial p_i}(\mathbf{q}) \lesssim \epsilon |\nabla \beta(\lambda^{-1}(\cdot - p))| + \frac{\lambda^3}{r^2(\lambda^2 + r^2)} + |\nabla \beta(\lambda^{-1}(\cdot - p))| \lambda^2 + \lambda^3 \quad (3.56)$$

and

$$|\epsilon d_{A(\mathbf{q})}^\epsilon [b(\mathbf{q}), \beta]| \lesssim (\epsilon \lambda^{-1} |\nabla(\lambda^{-1}(\cdot - p))| + \lambda^2 + \epsilon |I_{\lambda,p}^2|) |\beta| + \epsilon |\nabla \beta|, \quad (3.57)$$

one obtains

$$\begin{aligned} & \left| \epsilon \int_{B^4} (d_{A(\mathbf{q})}^\epsilon \alpha, [b(\mathbf{q}), \beta]) \right| = \left| \epsilon \int_{B^4} (\alpha, d_{A(\mathbf{q})}^* [b(\mathbf{q}), \beta]) \right| \\ & \lesssim \int_{B_{2\lambda}(p) \setminus B_\lambda(p)} \left(\epsilon^2 \lambda^{-1} + \epsilon \lambda + \lambda^4 + \frac{\epsilon^2 \lambda^2}{r(\lambda^2 + r^2)} + \frac{\epsilon \lambda^2}{r^2(\lambda^2 + r^2)} + \frac{\epsilon \lambda^4}{r(\lambda^2 + r^2)} \right) |\beta| dx \\ & \quad + \int_{B_{2\lambda}(p) \setminus B_\lambda(p)} (\epsilon^2 + \epsilon \lambda^2) |\nabla \beta| dx \\ & \quad + \int_{B^4 \setminus B_{\lambda/4}(p)} \left(\lambda^5 + \frac{\lambda^5}{r^2(\lambda^2 + r^2)} + \frac{\epsilon \lambda^5}{r^3(\lambda^2 + r^2)^2} + \frac{\epsilon \lambda^5}{r(\lambda^2 + r^2)} \right) |\beta| dx \\ & \quad + \int_{B^4 \setminus B_{\lambda/4}(p)} \left(\epsilon \lambda^3 + \frac{\epsilon \lambda^3}{r^2(\lambda^2 + r^2)} \right) |\nabla \beta| dx \\ & \lesssim (\epsilon^2 \lambda^2 + \epsilon \lambda^4 + \lambda^7 + \epsilon^2 \lambda^2 + \epsilon \lambda + \epsilon \lambda^4) \|\beta\|_{L^4(B_{2\lambda}(p) \setminus B_\lambda(p))} + (\epsilon^2 \lambda^2 + \epsilon \lambda^4) \|\nabla \beta\|_{L^2(B_{2\lambda}(p) \setminus B_\lambda(p))} \\ & \quad + (\lambda^5 + \lambda^4 + \epsilon \lambda + \epsilon \lambda^5 |\log \lambda|) \|\beta\|_{L^4(B^4 \setminus B_{\lambda/4}(p))} + (\epsilon \lambda^3 + \epsilon \lambda^2) \|\nabla \beta\|_{L^2(B^4 \setminus B_{\lambda/4}(p))} \\ & \lesssim \epsilon^{3/2} \|\beta\|_{A(\mathbf{q});1,2}. \end{aligned} \quad (3.58)$$

From (3.58), the first part of Lemma 3.5 (for $i = 1$) follows.

To prove the second part, for any given $\beta \in L_{0,1}^2(T^*B^4 \otimes \text{Ad}(P(\mathbf{q})))$ we estimate

$$\begin{aligned} & \langle (d_{A(\mathbf{q})}^\epsilon d_{A(\mathbf{q})}^* - d_{\tilde{A}(\mathbf{q})}^\epsilon d_{\tilde{A}(\mathbf{q})}^*) \mathbf{a}_i(\mathbf{q}), \beta \rangle = \int_{B^4} (d_{A(\mathbf{q})}^* \mathbf{a}_i(\mathbf{q}), d_{A(\mathbf{q})}^\epsilon \beta) dx - \int_{B^4} (d_{\tilde{A}(\mathbf{q})}^* \mathbf{a}_i(\mathbf{q}), d_{\tilde{A}(\mathbf{q})}^\epsilon \beta) dx \\ &= \int_{B^4} ((d_{A(\mathbf{q})}^* - d_{\tilde{A}(\mathbf{q})}^*) \mathbf{a}_i(\mathbf{q}), d_{A(\mathbf{q})}^\epsilon \beta) dx + \int_{B^4} (d_{\tilde{A}(\mathbf{q})}^* \mathbf{a}_i(\mathbf{q}), (d_{A(\mathbf{q})}^\epsilon - d_{\tilde{A}(\mathbf{q})}^\epsilon) \beta) dx \\ &= \int_{B^4 \setminus B_{\lambda/4}(p)} (\epsilon * [b(\mathbf{q}), * \mathbf{a}_i(\mathbf{q})], d_{A(\mathbf{q})}^\epsilon \beta) dx + \int_{B^4 \setminus B_{\lambda/4}(p)} (d_{\tilde{A}(\mathbf{q})}^* \mathbf{a}_i(\mathbf{q}), \epsilon * [b(\mathbf{q}), * \beta]) dx \\ & \lesssim \epsilon \left(\int_{B^4 \setminus B_{\lambda/4}(p)} |\mathbf{a}_i(\mathbf{q})|^2 dx \right)^{1/2} \|\beta\|_{A(\mathbf{q});1,2} + \epsilon \left(\int_{B^4 \setminus B_{\lambda/4}(p)} |d_{\tilde{A}(\mathbf{q})}^* \mathbf{a}_i(\mathbf{q})|^{4/3} dx \right)^{3/4} \|\beta\|_{A(\mathbf{q});1,2}. \end{aligned} \quad (3.59)$$

By (3.56), the first integral in (3.59) is estimated as

$$\int_{B^4 \setminus B_{\lambda/4}(p)} |\mathbf{a}_1(\mathbf{q})|^2 dx \lesssim \epsilon^2 \lambda^4 + \lambda^2 + \lambda^8 + \lambda^6 \lesssim \epsilon. \quad (3.60)$$

As for the second integral, since $|d_{\tilde{A}(\mathbf{q})}^* \epsilon \mathbf{a}_1| \leq |d^* \mathbf{a}_1| + \epsilon |[\tilde{A}(\mathbf{q}), * \mathbf{a}_1(\mathbf{q})]|$, from (3.12), (3.19), (3.44), (3.56), it follows that

$$\begin{aligned} |d_{\tilde{A}(\mathbf{q})}^* \epsilon \mathbf{a}_1(\mathbf{q})| &\lesssim \epsilon \lambda^{-1} |\nabla^2 \beta(\lambda(\cdot - p))| + \epsilon |\nabla \beta(\lambda(\cdot - p))| + \frac{\lambda^3}{r^3(\lambda^3 + r^2)} \\ &\quad + \lambda |\nabla^2 \beta(\lambda/4(\cdot - p))| + \lambda^2 |\nabla \beta(\lambda(\cdot - p))| + \lambda^3 \\ &\quad + \frac{\lambda^2}{r(\lambda^2 + r^2)} \left(\epsilon |\nabla \beta(\lambda(\cdot - p))| + \frac{\lambda^3}{r^2(\lambda^2 + r^2)} + \lambda^2 |\nabla \beta(\lambda/4(\cdot - p))| + \lambda^3 \right). \end{aligned} \quad (3.61)$$

Thus,

$$\begin{aligned} \int_{B^4 \setminus B_{\lambda/4}(p)} |d_{\tilde{A}(\mathbf{q})}^* \epsilon \mathbf{a}_1(\mathbf{q})|^{4/3} dx &\lesssim \epsilon^{4/3} \lambda^{8/3} + \epsilon^{4/3} \lambda^4 + \int_{\lambda/4}^2 \frac{\lambda^4}{r(\lambda^2 + r^2)^{4/3}} dr \\ &\quad + \lambda^{16/3} + \lambda^{20/3} + \lambda^4 + \epsilon^{4/3} \lambda^{8/3} \int_{\lambda}^{2\lambda} \frac{r^{5/3}}{(\lambda^2 + r^2)^{4/3}} dr + \lambda^{20/3} \int_{\lambda/4}^2 \frac{1}{r(\lambda^2 + r^2)^{8/3}} dr \\ &\quad + \lambda^{16/3} \int_{\lambda}^{2\lambda} \frac{r^{5/3}}{(\lambda^2 + r^2)^{4/3}} dr + \lambda^{20/3} \int_{\lambda/4}^2 \frac{r^{5/3}}{(\lambda^2 + r^2)^{4/3}} dr \\ &\lesssim \epsilon^{4/3} \lambda^{8/3} + \epsilon^{4/3} \lambda^4 + \lambda^{4/3} + \lambda^{16/3} + \lambda^{20/3} + \lambda^4 + \epsilon^{4/3} \lambda^{8/3} + \lambda^{4/3} + \lambda^{16/3} + \lambda^{20/3} |\log \lambda| \lesssim \epsilon^{2/3}. \end{aligned} \quad (3.62)$$

Combining (3.59), (3.60), (3.62), one obtains

$$|\langle (d_{A(\mathbf{q})} \epsilon d_{A(\mathbf{q})}^* \epsilon - d_{\tilde{A}(\mathbf{q})} \epsilon d_{\tilde{A}(\mathbf{q})}^* \epsilon) \mathbf{a}_i(\mathbf{q}), \beta \rangle| \lesssim \epsilon^{3/2} \|\beta\|_{A(\mathbf{q});1,2}, \quad (3.63)$$

i.e., the second assertion of Lemma 3.5 for $i = 1$. \square

Lemma 3.6 *Let $\mathbf{a}_i(\mathbf{q})$ be the elements of the orthonormal basis constructed in Lemma 3.2. The following estimates hold:*

$$\|\mathbf{a}_{i\mathbf{q}_j}(\mathbf{q})^\perp\|_{A(\mathbf{q});1,2;B^4} \lesssim \epsilon \quad (3.64)$$

for $1 \leq i, j \leq 8$, where $\mathbf{a}_{i\mathbf{q}_j}(\mathbf{q})$ denotes the directional derivative of $\mathbf{a}_i(\mathbf{q})$ in the direction \mathbf{q}_j .

Proof: We prove the lemma for $i = 1, j = 1$. The remaining cases are similar. For $A(\mathbf{q})$ as represented in §1, we first observe that

$$(\mathbf{a}_{1\mathbf{q}_1}(\mathbf{q}))^\perp = \left(a_{11}(\mathbf{q}) \frac{\partial}{\partial p_1} \left(a_{11}(\mathbf{q}) \frac{\partial A}{\partial p_1}(\mathbf{q}) \right) \right)^\perp = a_{11}(\mathbf{q})^2 \left(\frac{\partial^2 A}{\partial p_1^2}(\mathbf{q}) \right)^\perp$$

so, we need to estimate $\left\| a_{11}(\mathbf{q})^2 \frac{\partial^2 A}{\partial p_1^2}(\mathbf{q}) \right\|_{A(\mathbf{q});1,2}$.

On $B^4 \setminus B_{\lambda/4}(p)$,

$$\begin{aligned} \epsilon \frac{\partial^2 A}{\partial p_1^2}(\mathbf{q}) = & -\epsilon \lambda^{-2} \frac{\partial^2 \beta}{\partial x_1^2}(\lambda^{-1}(\cdot - p)) \underline{A}_\epsilon + g \frac{\partial^2 I_{\lambda,p}^2}{\partial p_1^2} g^{-1} + 16 \lambda^{-2} \frac{\partial^2 \beta}{\partial x_1^2}(4 \lambda^{-1}(\cdot - p)) g h_{\lambda,p} g^{-1} \\ & - 4 \lambda^{-1} \frac{\partial \beta}{\partial x_1}(4 \lambda^{-1}(\cdot - p)) g \frac{\partial h_{\lambda,p}}{\partial p_1} g^{-1} + (\beta_{\lambda/4,p} - 1) g \frac{\partial^2 h_{\lambda,p}}{\partial p_1^2} g^{-1}. \end{aligned} \quad (3.65)$$

Hence,

$$\begin{aligned} \left| \epsilon \frac{\partial^2 A}{\partial p_1^2}(\mathbf{q}) \right| \lesssim & \epsilon \lambda^{-2} |\nabla^2 \beta(\lambda^{-1}(\cdot - p))| + \frac{\lambda^2}{r^3(\lambda^2 + r^2)} + |\nabla^2 \beta(\lambda^{-1}(\cdot - p))| \\ & + \lambda |\nabla \beta(\lambda^{-1}(\cdot - p))| + \lambda^2, \end{aligned} \quad (3.66)$$

$$\begin{aligned} \left| \epsilon \nabla \left(\frac{\partial^2 A}{\partial p_1^2}(\mathbf{q}) \right) \right| \lesssim & \epsilon \lambda^{-3} |\nabla^3 \beta(\lambda^{-1}(\cdot - p))| + \epsilon \lambda^{-2} |\nabla^2 \beta(\lambda^{-1}(\cdot - p))| + \frac{\lambda^2}{r^4(\lambda^2 + r^2)} \\ & + \lambda^{-1} |\nabla^3 \beta(\lambda^{-1}(\cdot - p))| + |\nabla^2 \beta(\lambda^{-1}(\cdot - p))| + \lambda |\nabla \beta(\lambda^{-1}(\cdot - p))| + \lambda^2, \end{aligned} \quad (3.67)$$

and

$$\begin{aligned} \left| \epsilon \left[A(\mathbf{q}), \epsilon \frac{\partial^2 A}{\partial p_1^2}(\mathbf{q}) \right] \right| \lesssim & \epsilon^2 \lambda^{-2} |\nabla^2 \beta(\lambda^{-1}(\cdot - p))| + \frac{\epsilon \lambda^2}{r^3(\lambda^2 + r^2)} + \epsilon |\nabla^2 \beta(4 \lambda^{-1}(\cdot - p))| \\ & + \epsilon \lambda |\nabla \beta(\lambda^{-1}(\cdot - p))| + \epsilon \lambda^2 + \frac{\epsilon}{r(\lambda^2 + r^2)} |\nabla^2 \beta(\lambda^{-1}(\cdot - p))| \\ & + \frac{\lambda^4}{r^4(\lambda^2 + r^2)^2} + \frac{\lambda^2}{r(\lambda^2 + r^2)} |\nabla^2 \beta(4 \lambda^{-1}(\cdot - p))| \\ & + \frac{\lambda^3}{r(\lambda^2 + r^2)} |\nabla \beta(4 \lambda^{-1}(\cdot - p))| + \frac{\lambda^4}{r(\lambda^2 + r^2)}. \end{aligned} \quad (3.68)$$

From (3.66)–(3.68), using $|a_{11}(\mathbf{q})| \lesssim \epsilon^{3/2}$, one obtains

$$\begin{aligned} a_{11}(\mathbf{q})^4 \int_{B^4 \setminus B_{\lambda/4}(p)} \left| \frac{\partial^2 A}{\partial p_1^2}(\mathbf{q}) \right|^2 dx \lesssim & \epsilon^6 + \epsilon^4 \int_{B^4 \setminus B_{\lambda/4}(p)} \frac{\lambda^4}{r^6(\lambda^2 + r^2)^2} dx + \epsilon^4 \lambda^4 \\ & + \epsilon^4 \lambda^6 + \epsilon^4 \lambda^4 \lesssim \epsilon^3, \end{aligned} \quad (3.69)$$

$$\begin{aligned} a_{11}(\mathbf{q})^4 \int_{B^4 \setminus B_{\lambda/4}(p)} \left| \nabla \left(\frac{\partial^2 A}{\partial p_1^2}(\mathbf{q}) \right) \right|^2 dx \lesssim & \epsilon^6 \lambda^{-2} + \epsilon^6 + \epsilon^4 \int_{B^4 \setminus B_{\lambda/4}(p)} \frac{\lambda^4}{r^8(\lambda^2 + r^2)^2} dx + \epsilon^6 \lambda^2 \\ & + \epsilon^4 \lambda^4 \lesssim \epsilon^2, \end{aligned} \quad (3.70)$$

and

$$\begin{aligned} & \int_{B^4 \setminus B_{\lambda/4}(p)} \left| \epsilon \left[A(\mathbf{q}), a_{11}(\mathbf{q})^2 \frac{\partial^2 A}{\partial p_1^2}(\mathbf{q}) \right] \right|^2 dx \\ & \lesssim \epsilon^8 + \epsilon^6 \int_{B^4 \setminus B_{\lambda/4}(p)} \frac{\lambda^4}{r^6(\lambda^2 + r^2)^2} dx + \epsilon^6 \lambda^4 + \epsilon^6 \lambda^6 + \epsilon^6 \lambda^4 + \epsilon^6 \int_{B_{2\lambda}(p) \setminus B_{\lambda}(p)} \frac{1}{r^2(\lambda^2 + r^2)^2} dx \\ & + \epsilon^4 \int_{B^4 \setminus B_{\lambda/4}(p)} \frac{\lambda^8}{r^8(\lambda^2 + r^2)^4} dx + \epsilon^4 \int_{B_{\lambda/2}(p) \setminus B_{\lambda/4}(p)} \frac{\lambda^4}{r^2(\lambda^2 + r^2)^2} dx \\ & + \epsilon^4 \int_{B_{\lambda/2}(p) \setminus B_{\lambda/4}(p)} \frac{\lambda^6}{r^2(\lambda^2 + r^2)^2} dx + \epsilon^4 \int_{B^4 \setminus B_{\lambda/4}(p)} \frac{\lambda^8}{r^2(\lambda^2 + r^2)^2} dx \lesssim \epsilon^2. \end{aligned} \quad (3.71)$$

Combining (3.69)–(3.71) yields

$$\left\| a_{11}(\mathbf{q})^2 \frac{\partial^2 A}{\partial p_1^2}(\mathbf{q}) \right\|_{A(\mathbf{q});1,2;B^4 \setminus B_{\lambda/4}(p)} \lesssim \epsilon. \quad (3.72)$$

Since $\epsilon A(\mathbf{q}) = g I_{\lambda,p}^1 g^{-1}$ and $\epsilon \frac{\partial^2 A}{\partial p_1^2} = g \frac{\partial^2 I_{\lambda,p}^1}{\partial p_1^2} g^{-1}$ on $B_{\lambda/4}(p)$, one has

$$\begin{aligned} \left| a_{11}(\mathbf{q})^2 \frac{\partial^2 A}{\partial p_1^2}(\mathbf{q}) \right| &\lesssim \frac{\epsilon^2}{(\lambda^2 + r^2)^{3/2}}, \\ \left| \epsilon \left[A(\mathbf{q}), a_{11}(\mathbf{q})^2 \frac{\partial^2 A}{\partial p_1^2}(\mathbf{q}) \right] \right| &\lesssim \frac{\epsilon^2}{(\lambda^2 + r^2)^2}, \\ \left| \nabla \left(a_{11}(\mathbf{q})^2 \frac{\partial^2 A}{\partial p_1^2}(\mathbf{q}) \right) \right| &\lesssim \frac{\epsilon^2}{(\lambda^2 + r^2)^2}. \end{aligned}$$

Hence,

$$\int_{B_{\lambda/4}(p)} \left| a_{11}(\mathbf{q})^2 \frac{\partial^2 A}{\partial p_1^2}(\mathbf{q}) \right|^2 dx \lesssim \epsilon^4 \int_{B_{\lambda/4}(p)} \frac{1}{(\lambda^2 + r^2)^3} dx \lesssim \epsilon^3, \quad (3.73)$$

$$\int_{B_{\lambda/4}(p)} \left| \nabla \left(a_{11}(\mathbf{q})^2 \frac{\partial^2 A}{\partial p_1^2}(\mathbf{q}) \right) \right|^2 dx \lesssim \epsilon^4 \int_{B_{\lambda/4}(p)} \frac{1}{(\lambda^2 + r^2)^4} dx \lesssim \epsilon^2, \quad (3.74)$$

$$\int_{B_{\lambda/4}(p)} \left| \epsilon \left[A(\mathbf{q}), a_{11}(\mathbf{q})^2 \frac{\partial^2 A}{\partial p_1^2}(\mathbf{q}) \right] \right|^2 dx \lesssim \epsilon^2 \int_{B_{\lambda/4}(p)} \frac{1}{(\lambda^2 + r^2)^4} dx \lesssim \epsilon^2. \quad (3.75)$$

From (3.73)–(3.75), one obtains

$$\left\| a_{11}(\mathbf{q})^2 \frac{\partial^2 A}{\partial p_1^2}(\mathbf{q}) \right\|_{A(\mathbf{q});1,2;B_{\lambda/4}(p)} \lesssim \epsilon, \quad (3.76)$$

and, finally, from (3.72) and (3.76), the assertion of Lemma 3.6 follows, for the case $i = 1, j = 1$. The remaining cases can be proved similarly. \square

References

- [1] S. K. Donaldson, P. B. Kronheimer: *The geometry of four-manifolds*. Oxford University Press, Oxford (1990).
- [2] D. S. Freed, K. Uhlenbeck: *Instantons and 4-manifolds, second edition*. Springer-Verlag, New York-Berlin-Heidelberg (1991).
- [3] T. Isobe, A. Marini: On topologically distinct solutions of the Dirichlet problem for Yang-Mills connections. *Car. Var.* **5** (1997), 345–358.
- [4] T. Isobe, A. Marini: Small coupling limit and multiple solutions to the Dirichlet Problem for Yang-Mills connections in 4 dimensions - Part I.
- [5] T. Isobe, A. Marini: Small coupling limit and multiple solutions to the Dirichlet Problem for Yang-Mills connections in 4 dimensions - Part II.

- [6] A. Marini: Dirichlet and Neumann boundary problems for Yang-Mills connections. *Comm. Pure and Appl. Math.* **45** (1992), 1015–1050.